

# On “Bosonic, Fermionic and Mixed” Super-symmetric 2-Dimensional Integrable Models.

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## Abstract

It is shown that supersymmetric integrable models in two dimensions, both relativistic (i.e. super-Toda type theories) and non-relativistic (reductions of super-KP hierarchies) can be associated to general Poisson-brackets structures given by superaffinizations of any bosonic Lie or any super-Lie algebra.

This result allows enlarging the set of supersymmetric integrable models, which are no longer restricted to the subclass of superaffinizations of purely fermionic super-Lie algebras (that is admitting fermionic simple roots only).

**Introduction** Investigating the properties of two-dimensional integrable theories have become quite popular among high energy physicists during the last years. There are quite good reasons for that, among them we can mention the connections with string theory. It is clear by now that the non-relativistic integrable equations in  $(1+1)$  dimension of KdV or NLS type encode the properties of the discretized version of two-dimensional gravity (in single and multi-matrix models formulation).

On the other hand two-dimensional relativistic theories of Toda type, whose simplest example is provided by the Liouville equation, are also relevant in many respect; for instance in the Polyakov formulation of string theory the Liouville equation enters when dealing with non-critical strings. An even deeper connection results from the geometrical approach to string theory [1],[2]. This is related to and motivates some of the topics here discussed. Since however they have been elucidated in the talk given by D. Sorokin, let me skip this point.

The problem of constructing supersymmetric generalizations of integrable models is a very crucial one. The physical motivations are well-known, and even if no

discretized version of super-Riemann surfaces leading to supermatrix models has been worked out so far, there is a hope that one can bypass this step assuming as fundamental objects the superintegrable hierarchies themselves.

From a purely mathematical point of view the problem of classifying all supersymmetric integrable models is quite challenging because of new features not present in the purely bosonic case. I will just mention here that in the bosonic case the situation is well-understood. Even if some problems are still opened (e.g. possible relations between hierarchies produced in different ways), the general lines are clear: one starts with a given affine Lie algebra  $\mathcal{G}$ , then an integrable hierarchy can be produced either through hamiltonian reduction [3] or through coset construction. This is true both for non-relativistic hierarchies and for Toda models (in the latter case two copies of the affine Lie algebra should be taken, one for each chirality).

On the contrary the situation is different in the supersymmetric case; due to some simple argument which will be presented later, it was commonly believed [4] that the only affine Lie algebra out of which one could obtain supersymmetric integrable hierarchies were the  $N = 1$  affinization of the superLie algebras admitting a presentation in terms of fermionic simple roots only. For that reason only the integrable hierarchies obtained from such affine superalgebras have been considered in the literature.

In this talk I will show that the above argument can be easily overcome and that interesting supersymmetric integrable models can be obtained from  $N = 1$  affinizations of any bosonic Lie algebras, as well as any super-Lie algebra (regardless if the simple roots are purely fermionic or necessarily some bosonic simple roots are present). With an abuse of language we can call the latter supersymmetric integrable models either “bosonic” or respectively “mixed”. “Fermionic” supersymmetric integrable models are the previously known ones. Therefore “bosonic”, “mixed” or “fermionic” supersymmetries specify the sort of (affine-Lie algebra) super-Poisson bracket structure we have to deal with. In any of these cases the resulting supermodels have ordinary supersymmetric properties.

**The Matrix SuperKP Hierarchies** The starting point for a bosonic integrable hierarchy in the AKS framework is a matrix-type Lax operator  $\mathcal{L}$

$$\mathcal{L} = \frac{\partial}{\partial x} + J(x) + \Lambda \quad (1)$$

where  $J(x)$  denotes a set of currents valued in the semisimple finite Lie algebra  $\mathcal{G}$ . They give rise to an affine algebra  $\hat{\mathcal{G}}$  which provides (one of) the Poisson brackets structure of the underlying model.  $\Lambda$  is a constant element in the loop algebra

$$\tilde{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}(\lambda, \lambda^{-1}) \quad (2)$$

where  $\lambda$  is a spectral parameter.

If  $\Lambda$  has a regularity property, that is if under its adjoint action  $\tilde{\mathcal{G}}$  can be splitted into

$$\tilde{\mathcal{G}} = \tilde{\mathcal{K}} \oplus \tilde{\mathcal{M}} \quad (3)$$

where

$$\begin{aligned} \tilde{\mathcal{K}} &=_{def} Ker(ad_{\Lambda}) \\ \tilde{\mathcal{M}} &=_{def} Im(ad_{\Lambda}) \end{aligned} \quad (4)$$

and  $\tilde{\mathcal{K}}$  is abelian, while

$$[\tilde{\mathcal{K}}, \tilde{\mathcal{M}}] \subset \tilde{\mathcal{M}} \quad (5)$$

then, by a similarity transformation which is uniquely defined and iteratively computed order by order in negative powers of the spectral parameter  $\lambda$ , we can diagonalize  $\mathcal{L} \mapsto \hat{\mathcal{L}}$ .  $\hat{\mathcal{L}}$  is valued in the Cartan (abelian) subalgebra of  $\mathcal{G}$ :

$$\hat{\mathcal{L}} = \Lambda + \partial_x + J_{\alpha} h_{\alpha} + \sum_{k=1}^{\infty} \lambda^{-k} R_{k,\alpha} h_{\alpha} \quad (6)$$

The Cartan coefficients  $R_{k,\alpha}$  are hamiltonian densities, whose integrals are in involution, for our integrable hierarchy, the Poisson brackets structure being given by  $\hat{\mathcal{G}}$ .

For a generic Lie algebra  $\mathcal{G}$  there are many possible choices of a regular element  $\Lambda$  corresponding to different hierarchies, but for any Lie algebra at least two choices are always possible: *i)*  $\Lambda$  is a sum over the simple positive roots of  $\tilde{\mathcal{G}}$ , *ii)*  $\Lambda$  is given by  $\lambda H$  with  $H$  any given Cartan element of  $\mathcal{G}$ . The first choice corresponds to generalized KdV-type hierarchies (KdV is recovered for  $sl(2)$ ) while the second corresponds to generalized NLS-type hierarchies (standard NLS is obtained from  $sl(2)$ ).

Inami and Kanno [5] proved that, under some restrictions, the above construction can be applied to the supersymmetric case. When dealing with  $N = 1$  supersymmetry one introduces a superspace parametrized by the bosonic and grassmann coordinate  $x, \theta$  respectively and a fermionic derivative

$$D \equiv D_X = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \quad (7)$$

SuperKdV-type hierarchies can be produced from a matrix Lax operator  $\mathcal{L}$  just as in the bosonic case, while  $\mathcal{L}$  is now given by

$$\mathcal{L} = D_X + \sum_i \Psi(X) + \Lambda \quad (8)$$

Here  $\Psi(X)$  denotes  $N = 1$  supercurrents-valued on a (super-)Lie algebra whose Poisson brackets are the  $N = 1$  affinization of the given (super-)Lie algebra. Since we are dealing with generalized KdV hierarchies,  $\Lambda$  is given by the sum over the simple roots. It is already transparent from the above formula that, since  $D$  and  $\Psi$  are fermionic, for consistency  $\Lambda$  as well must be fermionic, which restricts the possible theories to those constructed from the superLie algebras which admit fermionic simple roots only. Indeed Inami and Kanno limited themselves to study this case.

The situation is clearly unsatisfactory, for instance one can ask what happens to supersymmetric NLS-type hierarchies: the regular element  $\Lambda$  should now be expressed by  $\Lambda = \lambda H$  ( $H$  in the Cartan) which is necessarily bosonic. The breaking of a definite statistics for the (eventual)  $\mathcal{L}$  operator in this case apparently suggests that either supersymmetric extensions of NLS-type hierarchies do not exist or that these ones cannot be systematically produced via the AKS framework. Both these statements prove to be uncorrect. Indeed, by other means, it has been shown [6, 7] that an integrable super-NLS hierarchy exists and moreover that admits as Poisson brackets structure the  $N = 1$  affinization of the (bosonic!)  $sl(2)$  algebra. Moreover it has a coset structure w.r.t. the  $N = 1$  affine  $U(1)$  subalgebra. This statement means that all the hamiltonian densities which provide the tower of hamiltonians in involution have vanishing Poisson brackets w.r.t. the above subalgebra.

At this point we have to understand if it is possible and how to fit such a result in the AKS framework. The ingredients have been given in [8]: it should be noticed the appearance of the spectral parameter  $\lambda$  in  $\Lambda = \lambda H$ . Since the Lax operator  $\mathcal{L}$  and its diagonalization are Laurent series in  $\lambda$  it makes sense and is indeed possible to introduce the notion of “alternated” or “twisted” bosonic or fermionic character of power series in  $\lambda$ ,

$$F(\lambda) = \sum_{k=-\infty}^{+\infty} \lambda^{2k} (\xi_k + \lambda \cdot \phi_k) \quad (9)$$

is an “alternated” fermion (boson) if  $\xi_k$  are fermionic while  $\phi_k$  are bosonic (and conversely). “Alternated” fermions and bosons have the same ring properties as ordinary bosons and fermions. It is clear at this point that we can assume  $\mathcal{L}$  being an “alternated” fermion and no contradiction with statistics will arise, for details see [8]. Notice that the theories produced out of this framework are ordinary supersymmetric theories in space and time since  $\lambda$  is only an auxiliary parameter.

**Hamiltonian reduction of any Super-WZNW model** For what concerns bosonic WZNW models, based on the Lie algebra  $\mathcal{G}$ , they are equivalent to two chiral copies  $J, \bar{J}$  of the affine  $\hat{\mathcal{G}}$  algebra

$$J = \partial g \cdot g^{-1}$$

$$\bar{J} = -g^{-1} \cdot \bar{\partial}g \quad (10)$$

( $g$  is valued in the group  $G$  based on  $\mathcal{G}$ ), satisfying the free equations of motion

$$\bar{\partial}J = \partial\bar{J} = 0 \quad (11)$$

The so-called abelian (hamiltonian) constrained bosonic WZNW model is obtained by setting the positive (negative) root component  $J_>$  ( $\bar{J}_<$ ) to satisfy

$$\begin{aligned} J_> &= \sum_i e_i \\ \bar{J}_< &= \sum_i f_i \end{aligned} \quad (12)$$

where the sums are over the positive (respectively negative) simple roots of  $\mathcal{G}$ .

By inserting the Gauß decomposition for  $g$  the constrained model is equivalent to a Toda field theory [3]. For  $sl(2)$  we get the Liouville equation.

One could think to repeat the same steps in the  $N = 1$  supersymmetric case as well. As before we deal with a superspace, a fermionic derivative  $D$ , and fermionic supercurrents defined as

$$\begin{aligned} \Psi &= -iDG \cdot G^{-1} \\ \bar{\Psi} &= iG^{-1}DG \end{aligned} \quad (13)$$

with  $G$  a supergroup element.

The free equations of the unconstrained model are

$$\bar{D}\Psi = D\bar{\Psi} = 0 \quad (14)$$

Since  $\Psi = \Psi_\alpha \tau^\alpha$  is fermionic,  $\tau^\alpha$  are the generators of the (super-)Lie algebra  $\mathcal{G}$ , then  $\Psi_\alpha$  have opposite statistics w.r.t. the corresponding generator in  $\mathcal{G}$ . It follows that in order to repeat the same steps as before to constrain the theory, we need to have a superalgebra admitting fermionic simple roots only. So for instance the standard superLiouville equation is recovered from the  $osp(1|2)$  algebra which admits a single fermionic simple root. Moreover such constraints turn out to be superconformal and, after gauge-fixing, the Dirac's brackets provide a super- $\mathcal{W}$  algebra (superVirasoro in case of  $osp(1|2)$ ). For that reason it is commonly believed [4] that constraining superWZNW from bosonic algebras or “mixed” superalgebras lead to non-supersymmetric models. Here again this statement proves wrong.

We can see this as follows [9]. Let us introduce a nilpotent Grassman differential

$$d \stackrel{\text{def}}{=} (dz - i\theta d\theta)\partial_z + d\theta D \quad (15)$$

(it can be easily checked that  $d^2 = 0$ ), we can introduce a Cartan form

$$\Omega =_{def} dG \cdot G^{-1} \quad (16)$$

which satisfy the Maurer-Cartan equation

$$d\Omega - \frac{1}{2}[\Omega, \Omega]_+ = 0 \quad (17)$$

where the anticommutator is understood in the Lie-algebraic context.

It follows that

$$\Omega = (dz - i\theta d\theta)J + id\theta\Psi \quad (18)$$

with

$$\begin{aligned} J &= J_\alpha \tau^\alpha =_{def} \partial G \cdot G^{-1} \\ \Psi &= \Psi_\alpha \tau^\alpha =_{def} -iD G \cdot G^{-1} \end{aligned} \quad (19)$$

As a consequence of the Maurer-Cartan equation satisfied by  $\Omega$  the  $J_\alpha$  superfields are not independent, but are constructed from the  $\Psi_\alpha$  superfields:

$$J = D\Psi - \frac{i}{2}[\Psi, \Psi]_+ \quad (20)$$

Let us specialize ourselves to the  $sl(2)$  case (the most general case, along the same lines, is treated in [10]). We are now in the position to constraint the composite supercurrents  $J_\alpha$  as before. Therefore we can set

$$J_- = 1 \quad (21)$$

which allows us imposing a further gauge-fixing

$$J_0|_{\theta=0} = 0 \quad (22)$$

Despite the fact that the above gauge-fixing is not manifestly supersymmetric it turns out to be indeed superconformal, for details see [9].

The above constraint and gauge-fixing can be explicitly solved in terms of the component fields entering the  $\Psi_i$  superfields: Let

$$\Psi_i = \xi_i(z) + \theta j_i(z) \quad (23)$$

(here  $i = 0, \pm$ ). In the  $sl(2)$  case we are left with 3 fundamental unconstrained fields, two fermionic and one bosonic, given by  $\xi_-$ ,  $\xi_+$  and  $j_+$ , with spin dimension respectively  $-(\frac{1}{2})$ ,  $\frac{3}{2}$  and 2.

The remaining fields are expressed through these ones.

Performing the analogous constraint for the second chirality and reexpressing  $\Psi$  through the superfields entering the Gauß decomposition of  $G$  we are led with a superconformal system of equations of motion

$$\bar{D}D\Phi = e^{2\Phi}\bar{\Psi}^+\Psi^-, \quad \bar{D}\Psi^- = 0 = D\bar{\Psi}^+; \quad (24)$$

$$D\Psi^- + 2D\Phi\Psi^- = 1, \quad \bar{D}\bar{\Psi}^+ + 2\bar{D}\Phi\bar{\Psi}^+ = 1. \quad (25)$$

In component fields we are led with the following system:

$$\begin{aligned} \square\phi &= e^{2\phi} \\ \bar{\partial}\psi &= 0 \\ \partial\bar{\psi} &= 0 \end{aligned} \quad (26)$$

with  $\psi, \bar{\psi}$  free fermions and  $\phi$  Liouville field. Such system as it can be checked is superconformal due to the nature of our constraints; the supersymmetry is realized non-linearly and spontaneously broken. Our system is based on a set of supersymmetric constraints. A peculiar feature is that the supersymmetric partner of a bosonic first class constraint is the second class. When analyzing the Dirac's brackets of the surviving fields we can prove they are equivalent to a Virasoro (spin 2 field) plus a free  $b - c$  system of weight  $(-\frac{1}{2}, \frac{3}{2})$ . The superconformal property of our model is reflected in the fact that there exists a Sugawara realization of the superVirasoro algebra in terms of these fields.

The fact that bosonic and fermionic fields are decoupled is a peculiar feature of the model based on  $sl(2)$ . It is not shared by more complicated models. In particular there is one which is rather interesting since it is based on the  $osp(1|4)$  algebra. This is the simplest superalgebra (the only at rank 2, see [11]) which admits a decomposition involving a simple fermionic and a simple bosonic root. This case is analyzed in [10].

Besides the nice mathematical properties of the above construction, the physical motivations are also quite important. More on that has been told by D. Sorokin.

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